Institute of Mechanics
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Modeling of Fiber-Reinforced Membrane Materials

MOOC@TU9

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Solutions – Week 2

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Solution of Task 1

In order to be able to use the same general solution for Task 1 and Task 2, the index "M" will not be used in the explanations regarding the first and second part of Task 1. In addition to that, the values of the specific parameters such as material constants or fiber angles are just inserted in part 3.

1. Since only two strain components are matter of interest, the matrix including the coefficients of the strain tensor $\boldsymbol{\varepsilon}$ is

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_\varphi & 0 \\ 0 & \varepsilon_z \end{pmatrix}. \tag{8}$$

Inserting this and the structural tensor

$$\boldsymbol{M}^{(a)} = \begin{pmatrix} \cos^2(\beta^{(a)}) & \cos(\beta^{(a)})\sin(\beta^{(a)}) \\ \cos(\beta^{(a)})\sin(\beta^{(a)}) & \sin^2(\beta^{(a)}) \end{pmatrix} \tag{9}$$

into the invariants (2), the expressions

$$J_1 = \varepsilon_\varphi + \varepsilon_z, \qquad J_2 = \varepsilon_\varphi^2 + \varepsilon_z^2 \qquad \text{and} \qquad J_4^{(a)} = \varepsilon_\varphi \cos^2(\beta^{(a)}) + \varepsilon_z \sin^2(\beta^{(a)}) \tag{10}$$

are obtained. Inserting them into equation (3) yields the energy function

$$\psi(\varepsilon_\varphi, \varepsilon_z) = \frac{\lambda}{2}\left(\varepsilon_\varphi + \varepsilon_z\right)^2 + \mu\left(\varepsilon_\varphi^2 + \varepsilon_z^2\right) + \sum_{a=1}^{2} \frac{\alpha^{(a)}}{2}\left(\varepsilon_\varphi \cos^2(\beta^{(a)}) + \varepsilon_z \sin^2(\beta^{(a)})\right)^2. \tag{11}$$

2. After computing the derivatives

$$\sigma_\varphi = \frac{\partial \psi}{\partial \varepsilon_\varphi} = \lambda\left(\varepsilon_\varphi + \varepsilon_z\right) + 2\mu\varepsilon_\varphi + \sum_{a=1}^{2} \alpha^{(a)}\left(\varepsilon_\varphi \cos^2(\beta^{(a)}) + \varepsilon_z \sin^2(\beta^{(a)})\right)\cos^2(\beta^{(a)}), \tag{12}$$

$$\sigma_z = \frac{\partial \psi}{\partial \varepsilon_z} = \lambda\left(\varepsilon_\varphi + \varepsilon_z\right) + 2\mu\varepsilon_z + \sum_{a=1}^{2} \alpha^{(a)}\left(\varepsilon_\varphi \cos^2(\beta^{(a)}) + \varepsilon_z \sin^2(\beta^{(a)})\right)\sin^2(\beta^{(a)}), \tag{13}$$

the system of equations can be formulated as

$$\sigma_\varphi = \left(\lambda + 2\mu + \sum_{a=1}^{2} \alpha^{(a)} \cos^4(\beta^{(a)})\right)\varepsilon_\varphi + \left(\lambda + \sum_{a=1}^{2} \alpha^{(a)} \sin^2(\beta_{(a)})\cos^2(\beta^{(a)})\right)\varepsilon_z, \tag{14}$$

$$\sigma_z = \left(\lambda + \sum_{a=1}^{2} \alpha^{(a)} \sin^2(\beta^{(a)})\cos^2(\beta^{(a)})\right)\varepsilon_\varphi + \left(\lambda + 2\mu + \sum_{a=1}^{2} \alpha^{(a)} \sin^4(\beta^{(a)})\right)\varepsilon_z. \tag{15}$$

3. Inserting the given fiber angles, equations (14) and (15) simplify to

$$\sigma_\varphi = \left(\lambda_\mathrm{M} + 2\mu_\mathrm{M} + \alpha_\mathrm{M}^{(1)}\right)\varepsilon_\varphi + \lambda_\mathrm{M}\varepsilon_z, \tag{16}$$

$$\sigma_z = \lambda_\mathrm{M}\varepsilon_\varphi + \left(\lambda_\mathrm{M} + 2\mu_\mathrm{M} + \alpha_\mathrm{M}^{(2)}\right)\varepsilon_z. \tag{17}$$

Note that here we consider the name index specifying the membrane material. Inserting $\varepsilon_z = 0$ and Barlow's formula $\sigma_\varphi = p_\mathrm{M} \frac{r_\mathrm{M}}{t_\mathrm{M}}$ into (16), (17) and solving for the unknown quantities yields

$$\varepsilon_\varphi = \frac{p_\mathrm{M}\, r_\mathrm{M}}{t_\mathrm{M}\left(\lambda_\mathrm{M} + 2\mu_\mathrm{M} + \alpha_\mathrm{M}^{(1)}\right)}, \tag{18}$$

$$\sigma_z = \lambda_\mathrm{M}\varepsilon_\varphi = \frac{p_\mathrm{M}\, r_\mathrm{M}\, \lambda_\mathrm{M}}{t_\mathrm{M}\left(\lambda_\mathrm{M} + 2\mu_\mathrm{M} + \alpha_\mathrm{M}^{(1)}\right)}. \tag{19}$$

Inserting the given values for the parameters leads to the resulting values

$$\varepsilon_\varphi = 1.25\,‰ < \varepsilon_{\varphi,\max} = 2\,\%,$$
$$\sigma_z = 41.1\,\mathrm{Pa} < \sigma_{z,\max} = 5\,\mathrm{kPa}.$$

Solution of Task 2

1. Due to the fact that the internal pressure is significantly higher and the fiber elasticity is significantly lower compared to the membrane material in Task 1, one could expect that huge strains may result here. As will be observed later this is not perfectly the case, although larger strains are obtained. One possible reason is the higher relative wall-thickness of the artery.
2. By inserting the given fiber angles for the artery into equations (14) and (15) we obtain

$$\sigma_\varphi = \left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{9}{8}\alpha_\mathrm{A}\right)\varepsilon_\varphi + \left(\lambda_\mathrm{A} + \frac{3}{8}\alpha_\mathrm{A}\right)\varepsilon_z, \tag{20}$$

$$\sigma_z = \left(\lambda_\mathrm{A} + \frac{3}{8}\alpha_\mathrm{A}\right)\varepsilon_\varphi + \left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{1}{8}\alpha_\mathrm{A}\right)\varepsilon_z. \tag{21}$$

   Utilizing $\varepsilon_z = 0$ and $\sigma_\varphi = p_\mathrm{A} \frac{r_\mathrm{A}}{t_\mathrm{A}}$ these equations can be solved for ε_φ and σ_z with the results

$$\varepsilon_\varphi = \frac{p_\mathrm{A}\, r_\mathrm{A}}{t_\mathrm{A}\left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{9}{8}\alpha_\mathrm{A}\right)}, \tag{22}$$

$$\sigma_z = \frac{p_\mathrm{A}\, r_\mathrm{A}\left(\lambda_\mathrm{A} + \frac{3}{8}\alpha_\mathrm{A}\right)}{t_\mathrm{A}\left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{9}{8}\alpha_\mathrm{A}\right)}. \tag{23}$$

   Numerical evaluation using the specific parameters for the artery yields

$$\varepsilon_\varphi = 13.80\,\% \qquad \text{and} \qquad \sigma_z = 56.30\,\mathrm{kPa}.$$

3. Using the arterial material for the membrane is equivalent to inserting the membrane geometry and pressure into equations (22) and (23):

$$\varepsilon_\varphi = \frac{p_\mathrm{M}\, r_\mathrm{M}}{t_\mathrm{M}\left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{9}{8}\alpha_\mathrm{A}\right)} = 215.7\,\%,$$

$$\sigma_z = \frac{p_\mathrm{M}\, r_\mathrm{M}\left(\lambda_\mathrm{A} + \frac{3}{8}\alpha_\mathrm{A}\right)}{t_\mathrm{M}\left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{9}{8}\alpha_\mathrm{A}\right)} = 879.65\,\mathrm{kPa}.$$

   The resulting circumferential strain is considerably high, thus, the aterial tissue cannot be used instead of the original membrane material!

4. If we wanted to use it however, we would have to increase the fiber stiffness in order to reduce the deformation. Solving

$$\varepsilon_\varphi = \frac{p_\mathrm{M}\, r_\mathrm{M}}{t_\mathrm{M}\left(\lambda_\mathrm{A} + 2\mu_\mathrm{A} + \frac{9}{8}\alpha_\mathrm{A}\right)} \le \varepsilon_\varphi^*$$

for α_A leads to

$$\alpha_\mathrm{A} = \alpha_\mathrm{A}^* \ge \frac{8}{9}\left(\frac{p_\mathrm{M}\, r_\mathrm{M}}{\varepsilon_\varphi^* t_\mathrm{M}} - \lambda_\mathrm{A} - 2\mu_\mathrm{A}\right).$$

The corresponding numerical value is

$$\alpha_\mathrm{A}^* \ge 7377\,\mathrm{kPa}.$$

Comparing this to the fiber stiffnesses of the original membrane material

$$\alpha_\mathrm{A}^* = 0.0037\,\alpha_\mathrm{M}^{(1)} = 0.0221\,\alpha_\mathrm{M}^{(2)},$$

shows that the required fiber stiffness of the arterial tissue is still lower than the ones of the membrane material.