Models in Civil Engineering: From Cardiology to Fishery

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Motivation

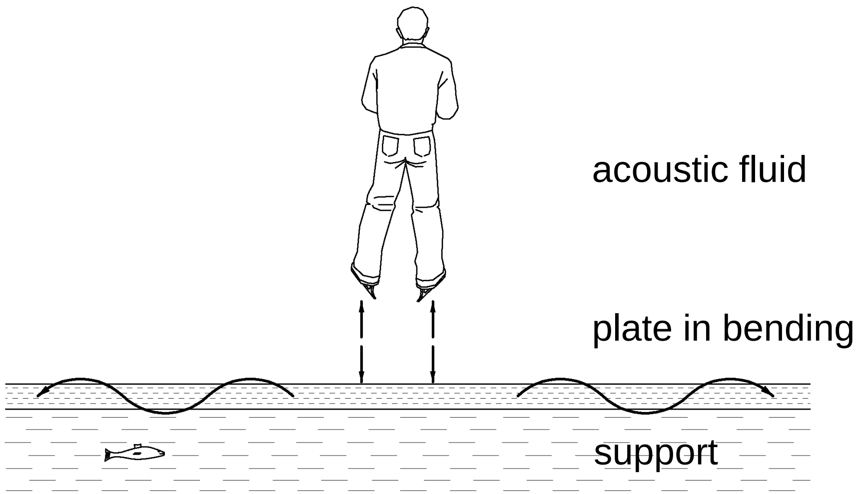
Can we “hear” the thickness of ice?

We need information about the following components:

- load
- ice layer
- sound field

Motivation

Ice layer under jump, plate under impulse loading



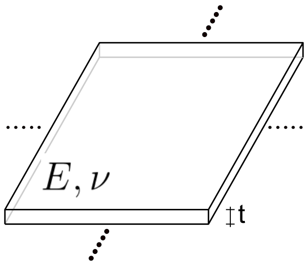
Can we hear the thickness of ice?

Content

1. Vibrations of beams and plates in bending
2. Wave propagation in a beam
   a) Wavelength, wavenumber
   b) Dispersion
3. Wave propagation in an acoustic fluid
4. Sound radiation
   a) Far field
   b) Evanescent field
5. Fourier analysis and spectral content of dynamic forces

Vibration of beams and plate in bending

Plate in bending action (neglecting the impedance of the underlying water)



B: bending stiffness $$B = \frac{Et^3}{12(1-\nu^2)}$$

Differential equation of a plate (considering the Kirchhoff theory):

$$B\left(\frac{\partial^4}{\partial x^4}w(x,y,t) + 2\frac{\partial^4}{\partial x^2 \partial y^2}w(x,y,t) + \frac{\partial^4}{\partial y^4}w(x,y,t)\right) + \mu \cdot \frac{\partial^2 w}{\partial t^2} = p(x,y,t)$$

PDE of 4th order

harmonic in time

Approach for solution

- Harmonically oscillating in space (x-,y- coordinate)
- Harmonically oscillating in time (t-coordinate)

} sin, cos

Excursus: Describing trigonometric functions with complex numbers (Euler‘s formula)

$$f(t) = A\cos(\Omega t) + B\sin(\Omega t) = \frac{1}{2}(A - iB)e^{i\Omega t} + \frac{1}{2}(A + iB)e^{i\Omega t}$$

two conjugate complex solutions

Vibration of beams and plate in bending

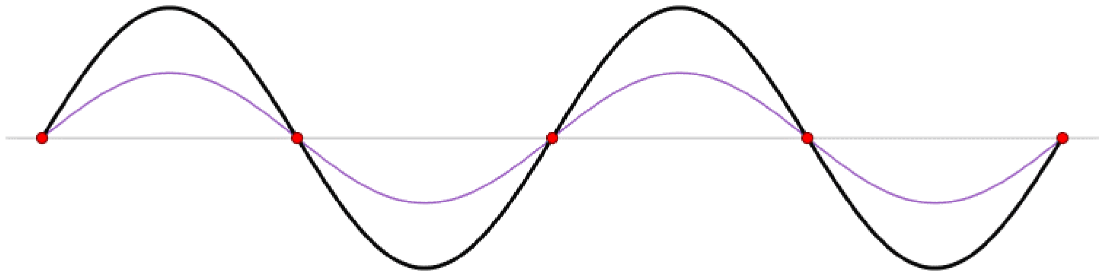
Plate in bending action

Approach for the computation of one part of the conjugate complex solution
(the complete solution is deduced easily)

$$w_+(x,y,t) = \hat{w}_+ e^{ik_x x} e^{ik_y y} e^{i\Omega t} \quad \text{with} \quad \begin{cases} k_x, k_y : & \text{wavenumbers} \\ \Omega : & \text{angular frequency} \end{cases}$$

$$\rightarrow \left[B(k_x^4 + 2k_x^2 k_y^2 + k_y^4) - \mu\Omega^2\right] \hat{w}_+ = p(k_x, k_y, \Omega)$$

$$\rightarrow \hat{w}_+ = \frac{p(k_x, k_y, \Omega)}{B(k_x^4 + 2k_x^2 k_y^2 + k_y^4) - \mu\Omega^2}$$



taken from wikipedia.org

Vibration of beams and plate in bending

Plate in bending action

Homogeneous solution

$$\rightarrow \left[\underbrace{B(k_x^4 + 2k_x^2 k_y^2 + k_y^4) - \mu\omega_e^2}_{\stackrel{!}{=}0} \right] \hat{w}_+^h = 0 \qquad (\underbrace{k_x^2 + k_y^2}_{k_B^2})^2 = \frac{\mu}{B}\omega_e^2$$

Therefore the wave propagation speed in dependence of the frequency can be calculated:

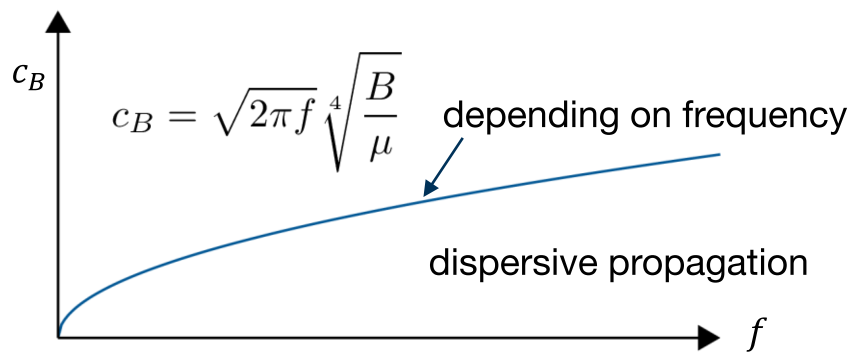
$$k_B^4 = \frac{\mu}{B}\omega_e^2 \qquad \rightarrow \qquad k_B = \sqrt[4]{\omega_e^2 \frac{\mu}{B}}$$

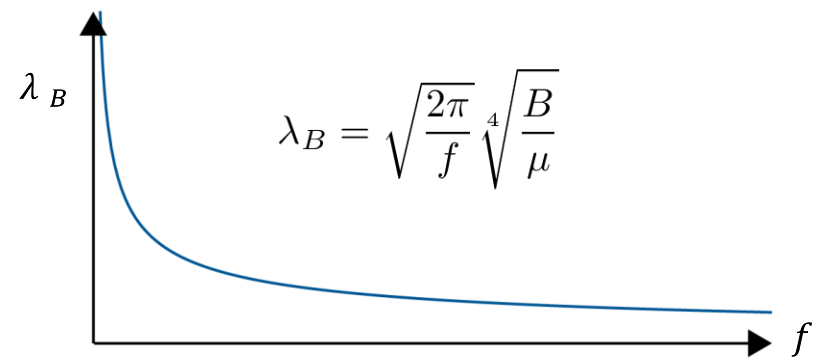
with $\boxed{k = \frac{2\pi}{\lambda}}$ $\qquad \lambda_B = \frac{2\pi}{\sqrt[4]{\omega_e^2 \frac{\mu}{B}}}$

with $\boxed{c = \lambda \cdot f}$ $\quad = \lambda \cdot \frac{\omega_e}{2\pi} \quad c_B = \sqrt[4]{\omega_e^2 \frac{B}{\mu}}$

Vibration of beams and plate in bending

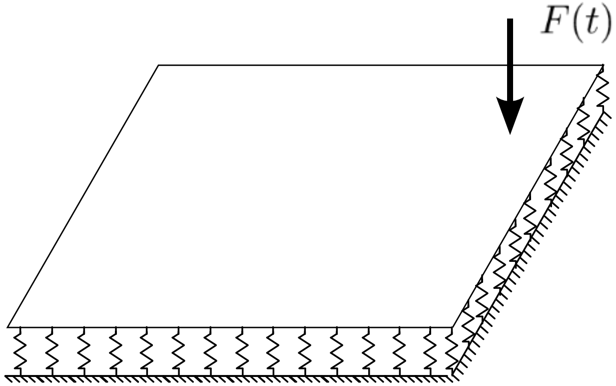
Visualisation: Dependency of the wave propagation speed and wavelength compared to the frequency





Vibration patterns of a finite plate under single load with different frequencies of excitation

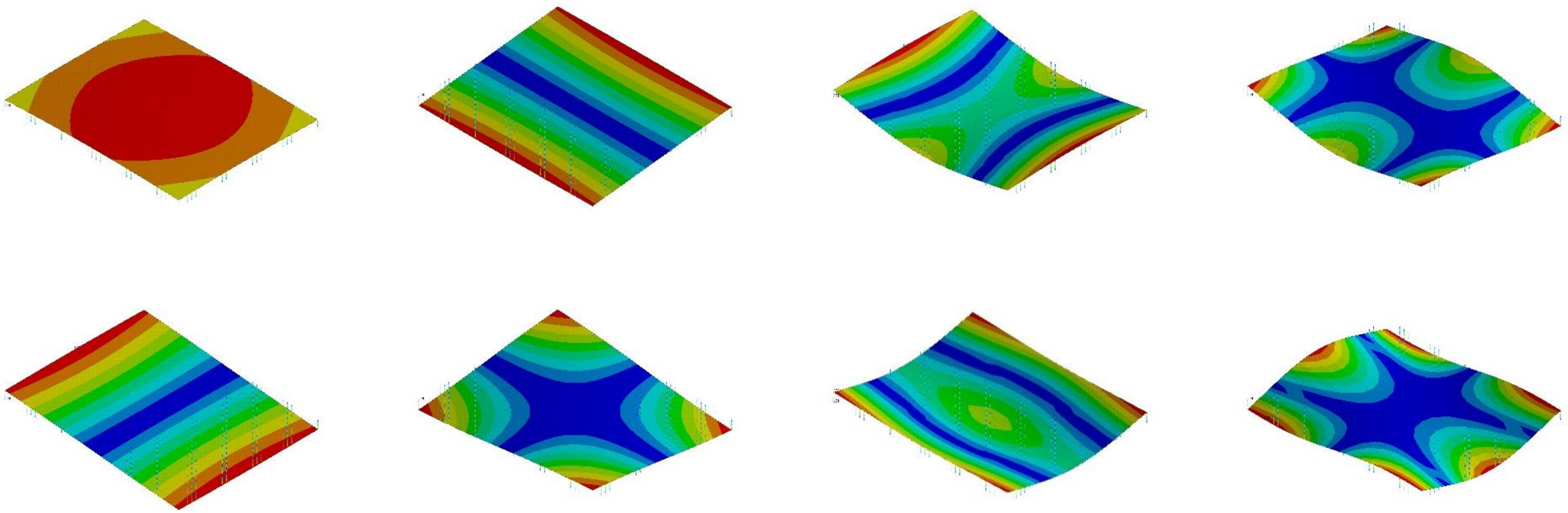
$$F(t) = F_0 \cdot \begin{cases} \sin \Omega_1 t \\ \sin \Omega_2 t \end{cases}$$



Vibration of beams and plate in bending

Plate in bending action

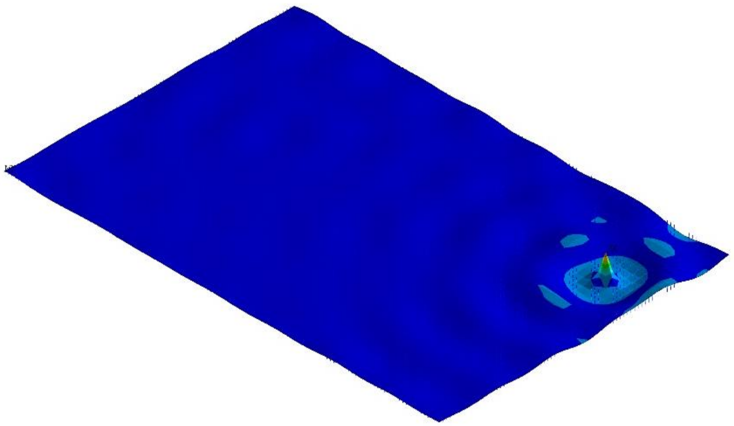
Computation results: Eigenmodes of an elastically supported plate



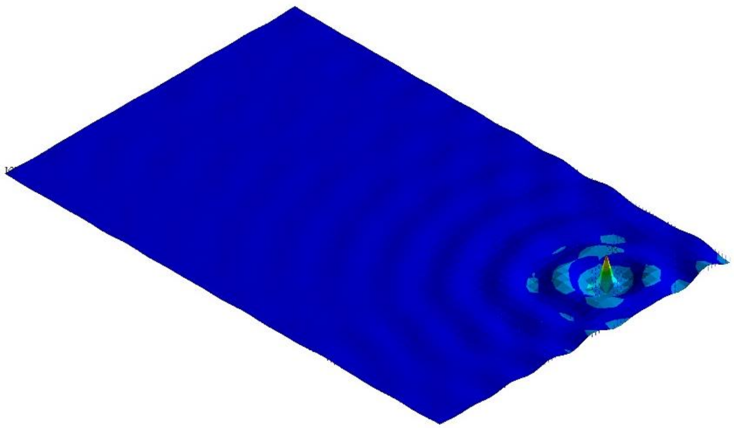
Vibration of beams and plate in bending

Plate in bending action

Computation results: Response to a harmonic single load in dependency of the frequency



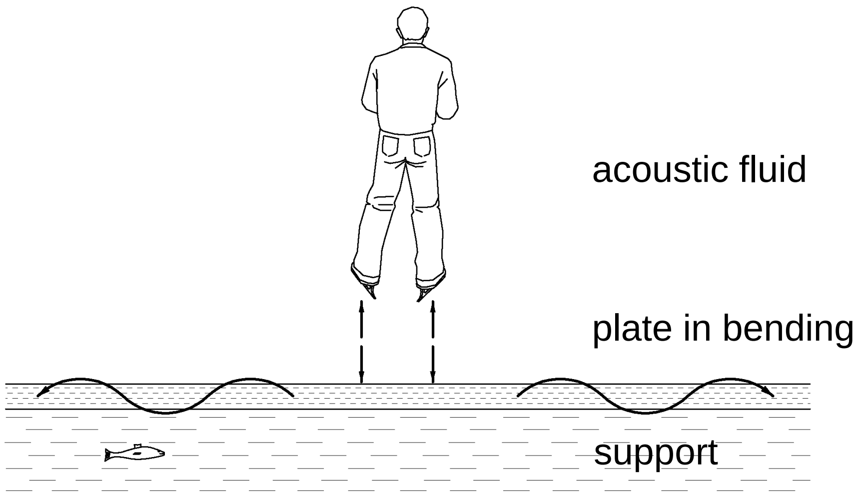
$\Omega_1 = 4.43\ Hz$



$\Omega_2 = 6.39\ Hz$

Motivation

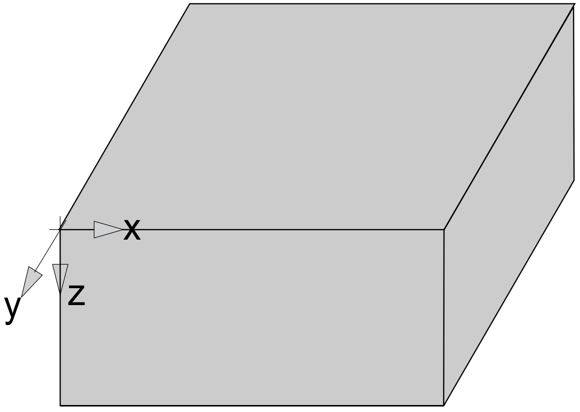
Ice layer under jump, plate under impulse loading



Acoustic fluid

Differential equation (wave equation)

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c_A^2}\frac{\partial^2 p}{\partial t^2} = 0$$ PDE of 2nd order



Approach:

$$p_+(x,y,z,t) = \hat{p}_+\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}\, e^{i\omega_e t}$$

$$\left(k_x^2 + k_y^2 + k_z^2 - \frac{\omega_e^2}{c_A^2}\right)\hat{p}_+ = 0 \qquad \rightarrow \qquad k_z = \sqrt{\frac{\omega_e^2}{c_A^2} - \left(k_x^2 + k_y^2\right)}$$

$$p_+(x,y,z,t) = \hat{p}_+ e^{ik_x x} e^{ik_y y} e^{i\sqrt{k_A^2 - \left(k_x^2 + k_y^2\right)}\, z} e^{i\omega_e t}$$

Acoustic fluid

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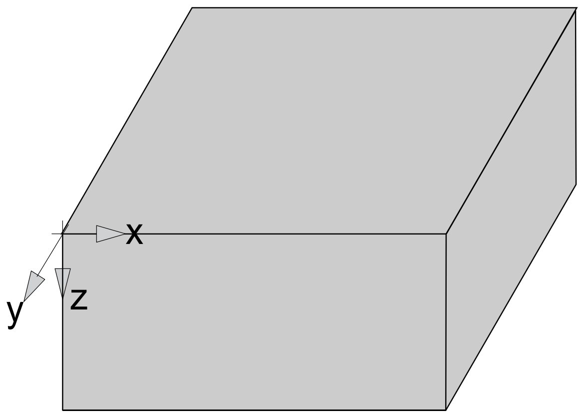

Approach:

$$p_+(x,y,z,t) = \hat{p}_+ \, e^{ik_x x} \, e^{ik_y y} \, e^{ik_z z} \, e^{i\omega_e t}$$

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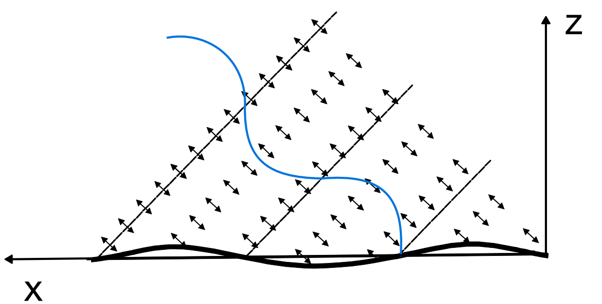
$$p_+(x,y,z,t) = \hat{p}_+ e^{ik_x x} e^{ik_y y} e^{i\sqrt{k_A^2 - \left(k_x^2 + k_y^2\right)}\,z} e^{i\omega_e t}$$

Acoustic fluid

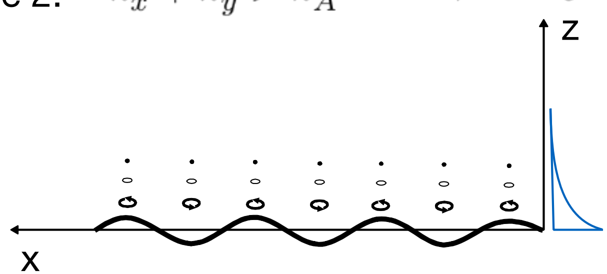
Discussion of the component of the sound-field, which is orthogonal to the x-y plane (z-coordinate)

Case 1: $k_x^2 + k_y^2 < k_A^2 \quad \rightarrow \quad e^{i\cdot\underbrace{\sqrt{|k_A^2 - (k_x^2 + k_y^2)|}}_{K}\cdot z} = \underbrace{\cos(K\,z) + i\,\sin(K\,z)}$



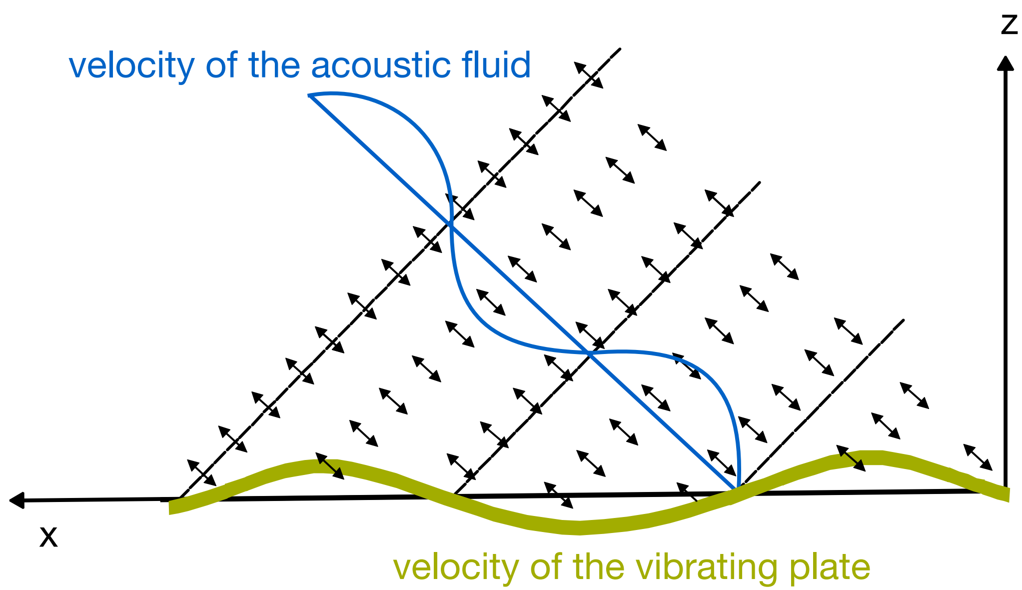
propagating wave
(far field)

Case 2: $k_x^2 + k_y^2 > k_A^2 \quad \rightarrow \quad e^{-\sqrt{|k_A^2 - (k_x^2 + k_y^2)|}\cdot z} = \underbrace{e^{-K\,z}}$



evanescent field

Coupling of plate and acoustic fluid



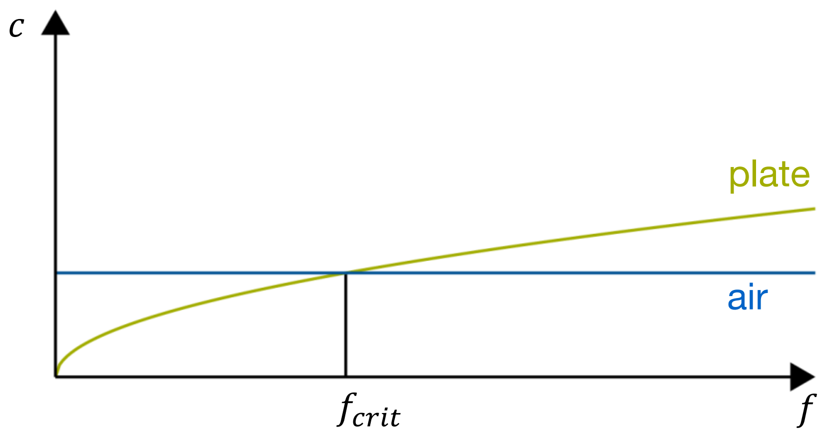
$\lambda_{plate} > \lambda_{air}$ in case of sound radiation (far field)

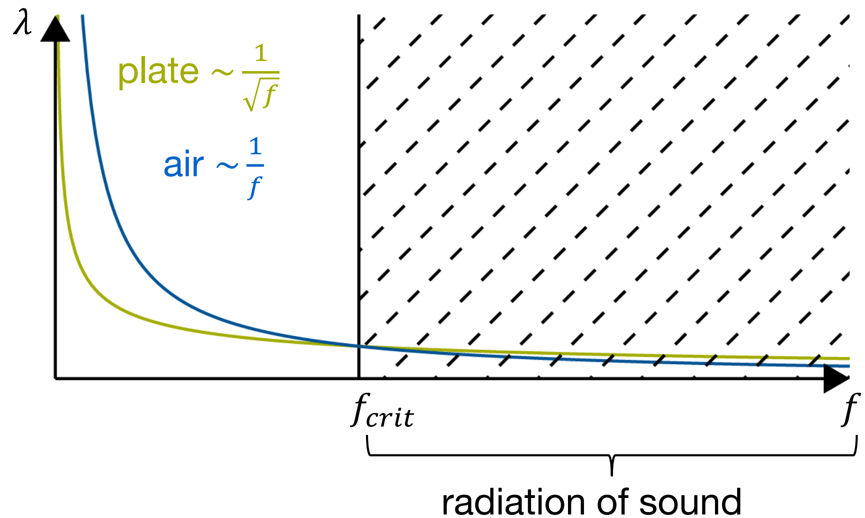
$\lambda_{plate} < \lambda_{air}$ in case of evanescent field (near field)

limit case: $\lambda_{plate} = \lambda_{air}$
$c_{plate} = c_{air}$ $\Rightarrow f_{crit}$

Coupling of plate and acoustic fluid

Velocity and wave length in dependency of frequency



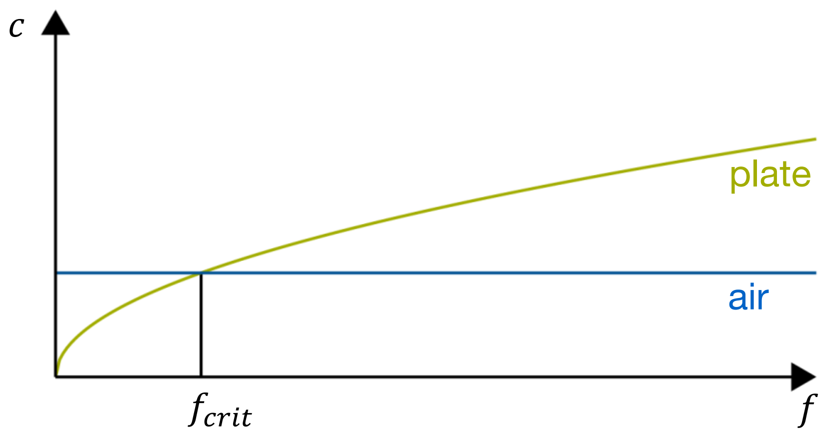


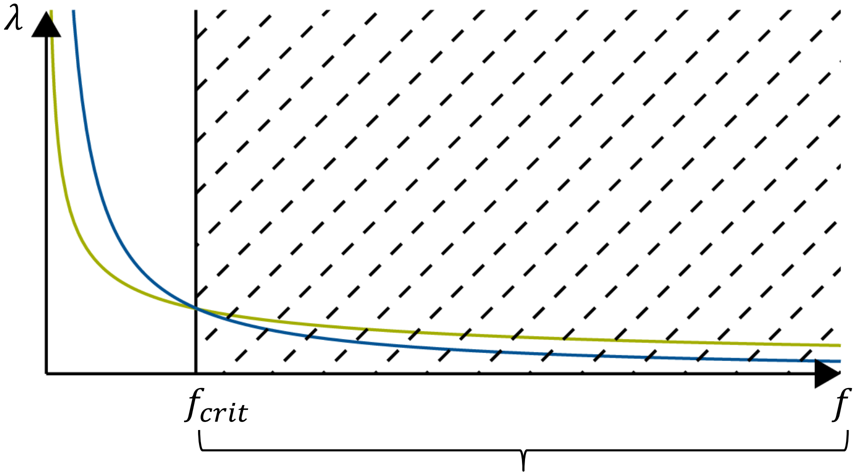
Radiation of sound is (in case of infinite layers) just possible for frequencies of excitation with $f > f_{crit}$

→ Investigation of the frequency content of the load necessary

Coupling of plate and acoustic fluid

Velocity and wave length in dependency of frequency





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Coupling of plate and acoustic fluid

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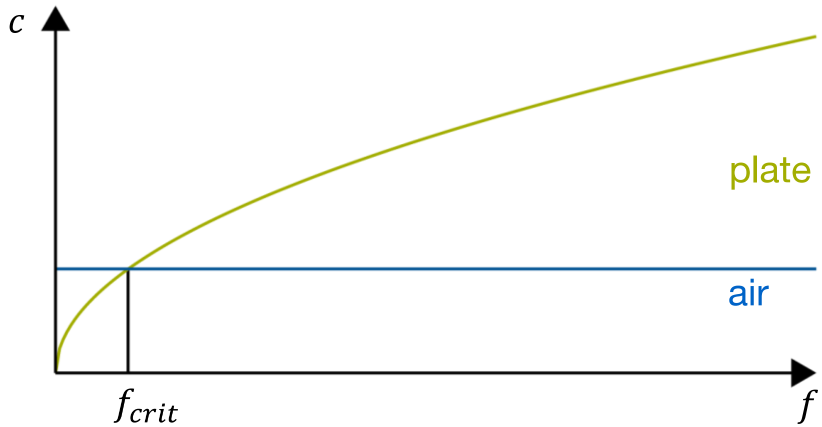
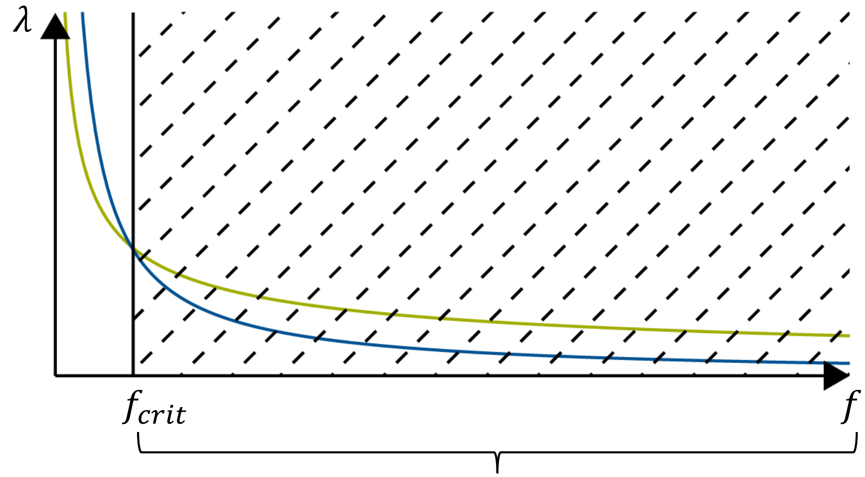
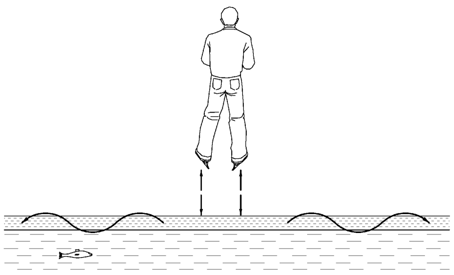




Radiation of sound is (in case of infinite layers) just possible for frequencies of excitation with $f > f_{crit}$

→ Investigation of the frequency content of the load necessary

Load



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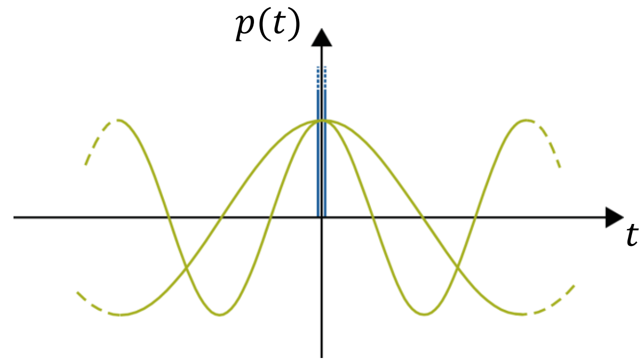
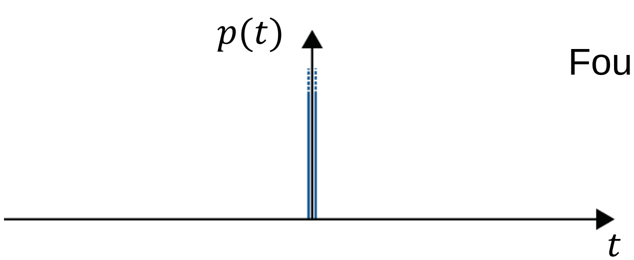


Remember: Fourier Series

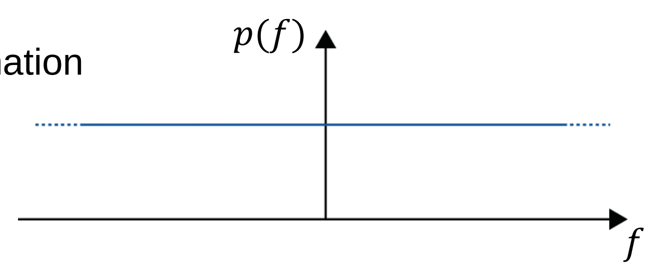
$$p(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{\frac{ik\pi t}{T}} \qquad \text{with:} \qquad c_k = \frac{1}{2T} \int_{-T}^{T} p(t) \cdot e^{-\frac{ik\pi t}{T}} dt$$

For an impulse an infinite number of cosine-members is needed



Fourier Transformation

○—●



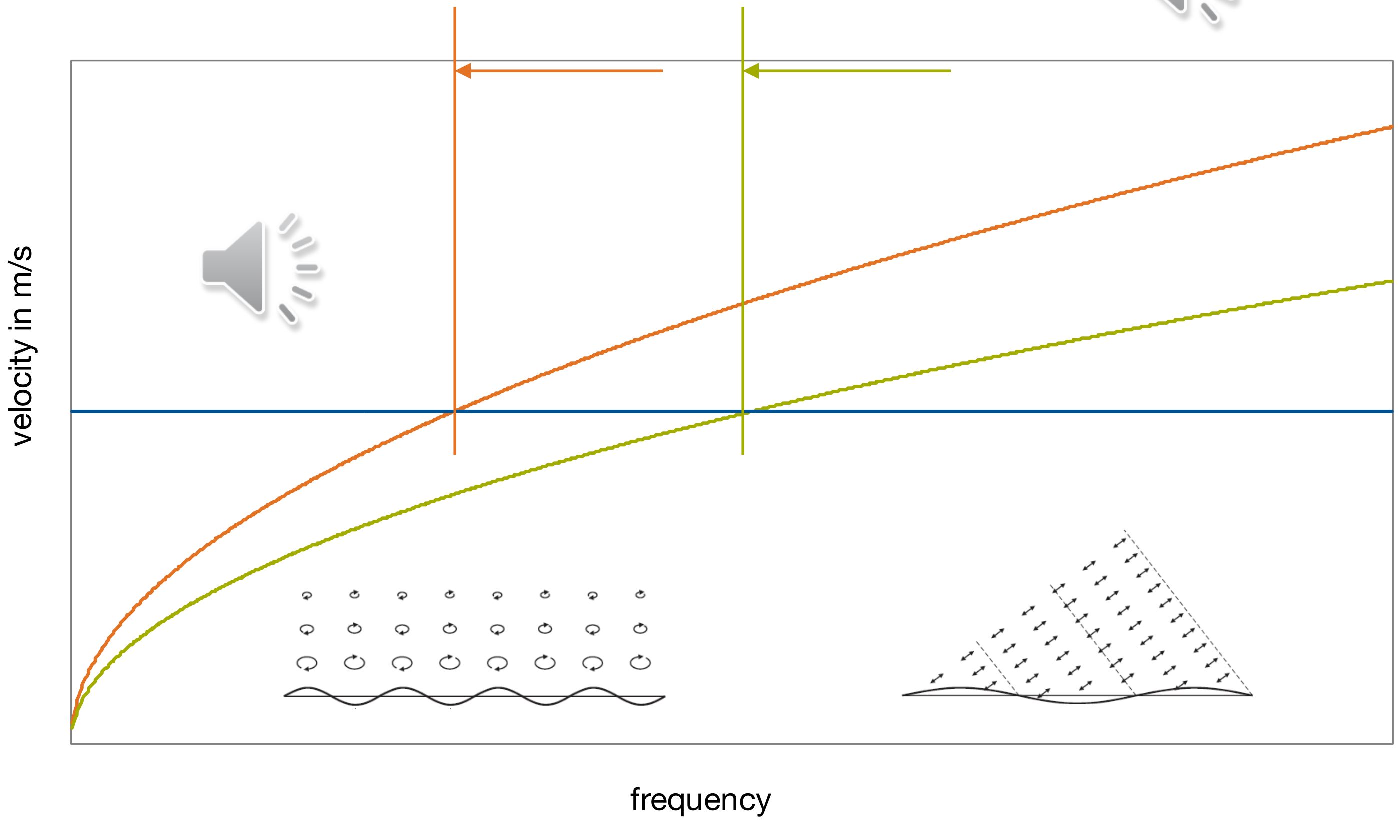
Conclusion

- All frequencies are part of the signal.
- Just $f > f_{crit}$ radiate sound (and can be heard).
- High frequencies propagate faster (bending waves are dispersive) and can be perceived earlier at different locations.
- The lowest frequency to be heard is f_{crit}.
- f_{crit} is depending on the thickness of the ice layer.

We can „hear the thickness of an ice layer“.

Example

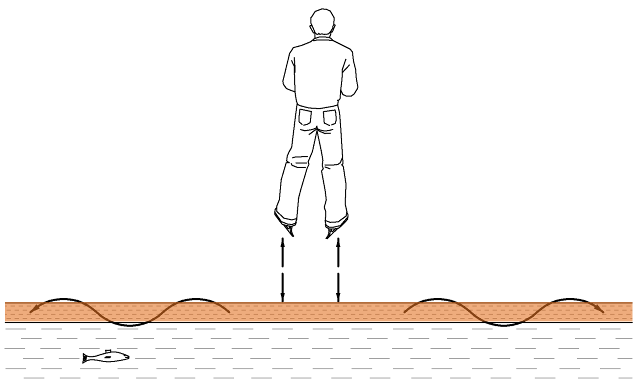
Velocity in dependency of frequency



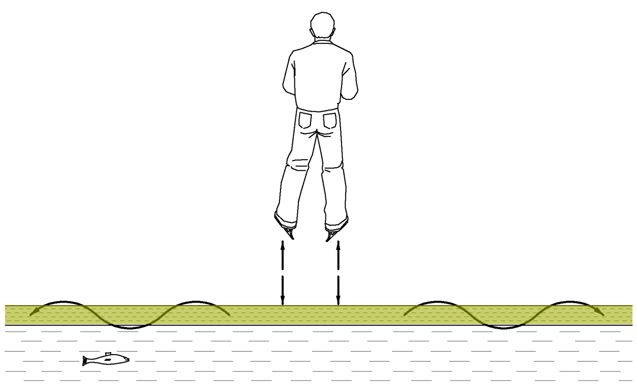
Task of the Week

Given: Results of acoustic measurements for two different ice layers

Layer 1



Layer 2

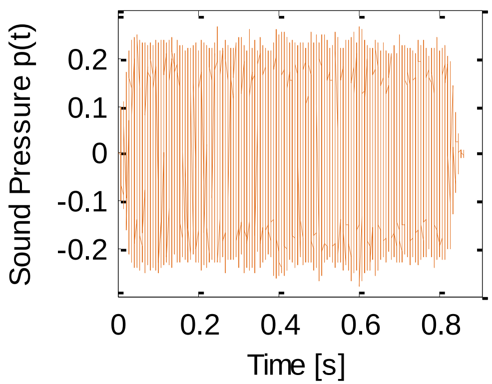


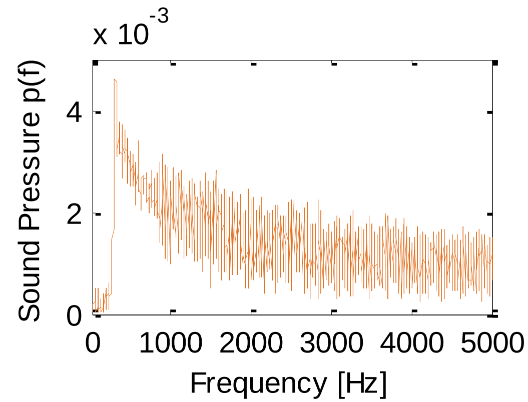
Task of the Week

Given: Results of acoustic measurements for two different ice layers

Time Domain | Frequency Domain [0Hz; 5kHz]

Layer 1





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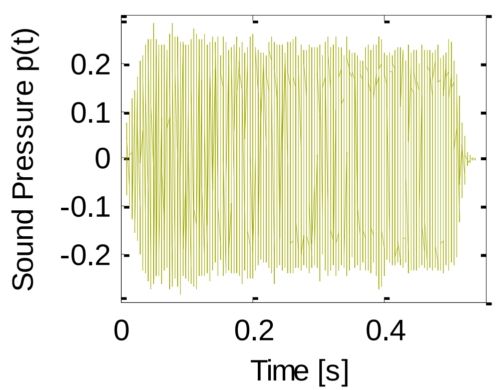

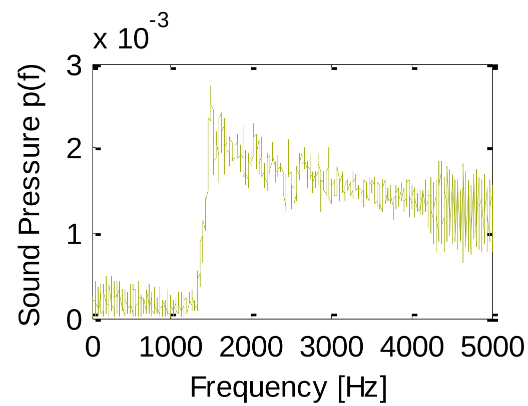

Task of the Week

Given: Results of acoustic measurements for two different ice layers

- Compute the thickness of the ice layers (Would you jump on the ice?)
  Use the following material parameters for ice
  - Young's Modulus $E = 9.1 \cdot 10^9 \, \frac{N}{m^2}$
  - Density $\rho = 916.7 \, \frac{kg}{m^3}$
  - Poisson's Ratio $\nu = 0.33$

  Use the following constant (frequency independent) wave velocity for air
  - Wave velocity $c_{air} = 340 \frac{m}{s}$

- Is the applied model conservative (on the save side)? Please compare your results with the results published by Lundmark (2001) and comment on the differences. Under what circumstances would you trust the results in order to jump on the ice?

- Estimate the stiffness of the water using the model of an elastically supported plate. Comment on the model of an elastic support of the plate.

Published Results

Lundmark, 2001