Institute of Mechanics
and Shell Structures
Faculty Civil Engineering
Chair of Mechanics

Modeling of Fiber-Reinforced Membrane Materials

MOOC@TU9

Daniel Balzani

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Tasks – Week 2

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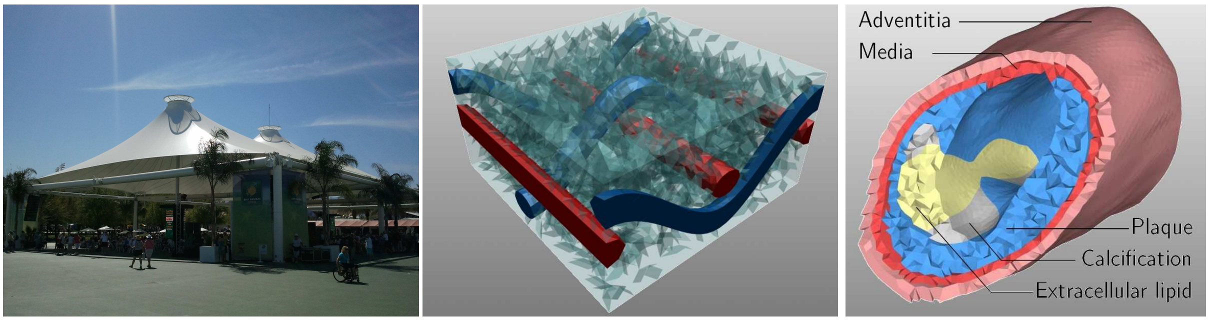
Technische Universität Dresden
Institut für Mechanik und Flächentragwerke
Prof. Dr.-Ing. habil. D. Balzani
Nürnberger Str. 31a
01187 Dresden
daniel.balzani@tu-dresden.de

Introduction

In civil engineering textile materials are an important class of materials in the context of light-weight constructions, cf. Fig. 1 (a). These materials are thin membranes and they consist of woven fiber networks embedded in a matrix, see Fig. 1 (b). Therefore, textile membrane materials are composite materials. Due to the woven nature of the networks these fiber fabrics are decomposed into the warp and fill fibers. Owing to the stiffer fibers (compared to the matrix) these membranes are able to resist rather high normal loads but very few bending loading. That is why these materials are used in conditions where the loading is rather biaxial and thus, they are mostly used as roof or facade construction elements. This material composition is conceptionally similar to soft biological tissues since there also collagen fibers and smooth muscle cells are embedded in an isotropic ground substance. In arterial walls for instance mainly two fiber families are observed which are arranged cross-wise helically around the artery. The biological fiber network is however not woven and thus the two orientations of the fiber families are not perpendicular to each other as for textile membrane materials.



(a) (b) (c)

Figure 1: (a) Textile membrane roof construction at the ATP tournement in Indian Wells, (b) material composition of membrane material and (c) schematic illustration of atherosclerotic artery.

In this exercise, fiber-reinforced structures are going to be analyzed. In detail a textile membrane roof construction and an arterial wall segment are considered and compared with each other. Here, we focus on simplified structures and a idealized description of the material behavior in order to match the state of knowledge of students with an engineering bachelor degree. Thus, we concentrate on thin membranes that do not resist bending moments although for instance arterial walls are typically thick-walled tubes, in particular diseased arteries, see Fig. 1 (c). With respect to the material modeling approach we focus on the small strain framework. In the context of continuum mechanical descriptions at small strains the stress tensor is computed by the derivative of a strain energy function $\psi(\boldsymbol{\varepsilon})$ with respect to the classical strain tensor $\boldsymbol{\varepsilon} = \text{Grad}^{\text{sym}}\boldsymbol{u}$, i.e.

$$\boldsymbol{\sigma} = \frac{\partial\psi(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}. \tag{1}$$

A specific function for the strain energy is constructed such that the resulting stress-stretch behavior corresponds with experimental data. This function is typically formulated in terms of coordinate-invariant quantities, the so-called invariants, which are functions of the strain tensor. For the representation of transverse isotropy, which reflects the symmetry of one fiber-reinforcement, a structural tensor $\boldsymbol{M}^{(a)} = \boldsymbol{a}^{(a)} \otimes \boldsymbol{a}^{(a)}$ is taken into account; $\boldsymbol{a}^{(a)}$ denotes the direction of fiber family (a). Then, a reasonable set of invariants is given by

$$J_1 = \text{tr}[\boldsymbol{\varepsilon}], \qquad J_2 = \text{tr}[\boldsymbol{\varepsilon}^2] \qquad \text{and} \qquad J_4^{(a)} = \text{tr}[\boldsymbol{\varepsilon}\boldsymbol{M}^{(a)}]. \tag{2}$$

The trace operator $\text{tr}[\boldsymbol{\varepsilon}]$ is defined as the sum of the main diagonal elements of the coefficient matrix of the strain tensor.

For the description of materials with mainly two fiber reinforcements the strain energy

$$\psi = \underbrace{\frac{1}{2}\,\lambda\,J_1^2 + \mu\,J_2}_{\psi^{\text{iso}}} + \sum_{a=1}^{2} \underbrace{\left[\frac{1}{2}\,\alpha^{(a)}\,\left(J_4^{(a)}\right)^2\right]}_{\psi^{\text{ti}}_{(a)}} \tag{3}$$

can then be considered. The first part ψ^{iso} refers to the isotropic matrix material and contains the Lamé constants λ and μ which are determined by the Young's modulus E and the Poisson ratio ν according to $\lambda = E\nu/[(1+\nu)\,(1-2\nu)]$ and $\mu = E/[2\,(1+\nu)]$. The response of fiber family (a) is described by the transverse isotropic energy $\psi^{\text{ti}}_{(a)}$, wherein $\alpha^{(a)}$ is a material parameter representing the elasticity of the fiber.

Additional remarks for the solution of the tasks:

Here, we focus on polar coordinates and then tensors are mainly defined by their coefficient matrix. For the structural tensor this means that the components of the coefficient matrix denoted by $\boldsymbol{M}^{(a)}$ are given by

$$M_{ij}^{(a)} = a_i^{(a)} a_j^{(a)}, \qquad i,j \in [1,2,3], \tag{4}$$

wherein $a_i^{(a)}$ are the coefficients of the fiber orientation vectors $\boldsymbol{a}^{(a)}$. For membrane structures with small thickness t we consider a two-dimensional idealization at each material point. There, the local coordinate system only consists of two base vectors $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_z$, cf. Fig. 2. In this local coordinate system the fiber orientation vector is computed by

$$\boldsymbol{a}^{(a)} = \begin{pmatrix} \cos(\beta^{(a)}) \\ \sin(\beta^{(a)}) \end{pmatrix}, \qquad a \in [1,2], \tag{5}$$

where $\beta^{(a)}$ denotes the angle between the circumferential direction φ and the fiber direction $\boldsymbol{a}^{(a)}$. Then the coefficient matrix of the structural tensor reads

$$\boldsymbol{M}^{(a)} = \begin{pmatrix} \cos^2(\beta^{(a)}) & \cos(\beta^{(a)})\sin(\beta^{(a)}) \\ \cos(\beta^{(a)})\sin(\beta^{(a)}) & \sin^2(\beta^{(a)}) \end{pmatrix}. \tag{6}$$

Due to the negligible stresses in thickness direction of the membrane, only the two-dimensional counterparts of the coefficients of the stress tensor have to be considered. Thus, the stresses in the local coordinate system are given by

$$\sigma_{ij} = \frac{\partial\psi(\varepsilon_{ij})}{\partial\varepsilon_{ij}} \quad \text{with} \quad i,j \in [\varphi, z]. \tag{7}$$

Note that for the tasks in this exercise only rotation-symmetric problems are considered such that in polar coordinates only homogeneous problems occur. In addition to that, in the specific structural problems no shear stresses occur in the given coordinates.

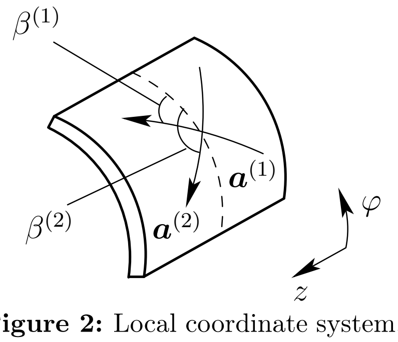


Figure 2: Local coordinate system.

Task 1: Textile membrane of a lightweight structure

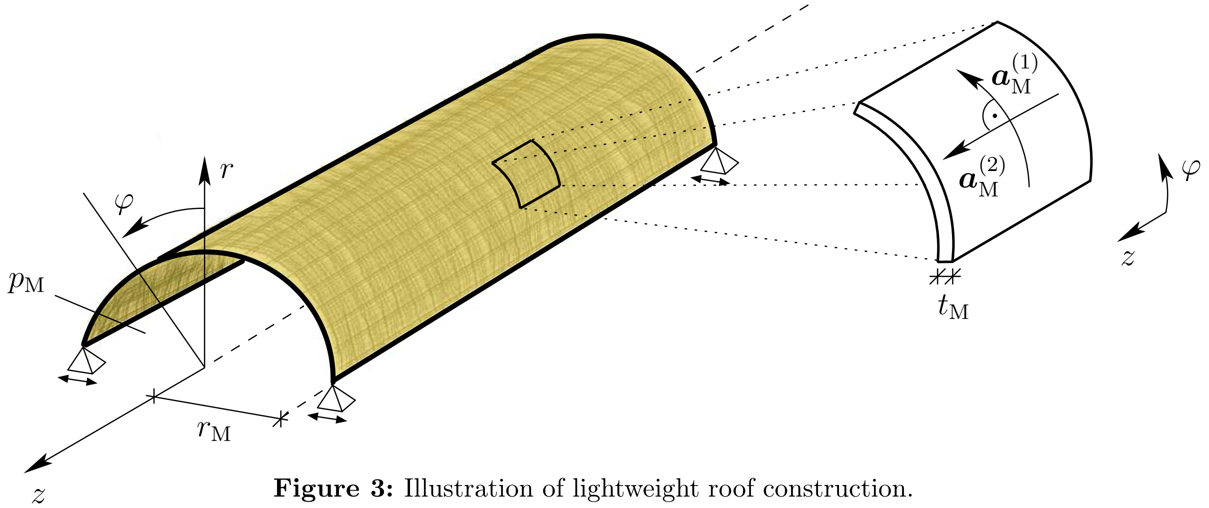


Figure 3: Illustration of lightweight roof construction.

Fig. 3 shows the idealized model of a lightweight roof construction which is inflated by an internal pressure p_{M}. The membrane material is woven and shows a different stiffness in the two different fiber directions. The stiffer warp direction is oriented in circumferential and the filling thread in axial direction. The given geometric variables, the internal pressure and the material parameters are

$r_{\mathrm{M}} = 10\,\mathrm{m}$,	$E_{\mathrm{M}} = 2\,\mathrm{kPa}$,	$\alpha_{\mathrm{M}}^{(2)} = \frac{1}{6}\alpha_{\mathrm{M}}^{(1)}$,
$t_{\mathrm{M}} = 1\,\mathrm{mm}$,	$\nu_{\mathrm{M}} = 0.49$,	$\beta_{\mathrm{M}}^{(1)} = 0°$,
$p_{\mathrm{M}} = 0.25\,\mathrm{kPa}$,	$\alpha_{\mathrm{M}}^{(1)} = 2 \cdot 10^6\,\mathrm{kPa}$,	$\beta_{\mathrm{M}}^{(2)} = 90°$.

The index "M" indicates the membrane material. Note that in the following double indices are denoted by one index only, i. e. $\sigma_\varphi := \sigma_{\varphi\varphi}$, $\sigma_z := \sigma_{zz}$, $\sigma_r := \sigma_{rr}$, $\varepsilon_\varphi := \varepsilon_{\varphi\varphi}$, $\varepsilon_z := \varepsilon_{zz}$, $\varepsilon_r := \varepsilon_{rr}$.

Due to the special structure and the idealization as a membrane, the stress components in radial direction as well as the shear stresses can be neglected and accordingly, the shear strains are zero. For simplicity, also the strains in thickness direction are assumed to vanish and one obtains

$$\sigma_r = \sigma_{r\varphi} = \sigma_{\varphi r} = \sigma_{rz} = \sigma_{zr} = \sigma_{\varphi z} = \sigma_{z\varphi} = 0,$$
$$\varepsilon_r = \varepsilon_{r\varphi} = \varepsilon_{\varphi r} = \varepsilon_{rz} = \varepsilon_{zr} = \varepsilon_{\varphi z} = \varepsilon_{z\varphi} = 0.$$

1. Determine the specific strain energy function by evaluating the invariants (2) for the specific structural tensor and inserting them into equation (3). (Hint: Do not yet insert the values for the parameters.)

2. Identify the expressions for the non-zero stresses by calculating the derivatives of the strain energy function, i. e. $\sigma_\varphi = \partial\psi/\partial\varepsilon_\varphi$ and $\sigma_z = \partial\psi/\partial\varepsilon_z$. This yields a linear system of equations consisting of two equations and the four unknowns σ_φ, σ_z, ε_φ and ε_z.

3. Due to the boundary conditions a plane strain state can be assumed in the r-φ-plane such that $\varepsilon_z = 0$. Furthermore, the circumferential stress σ_φ can be eliminated using Barlow's formula

$$\sigma_\varphi = p_{\mathrm{M}}\,\frac{r_{\mathrm{M}}}{t_{\mathrm{M}}}.$$

   Calculate the values for the remaining unknown quantities ε_φ and σ_z and verify, if the ultimate values $\varepsilon_{\varphi,\max} = 2\,\%$ and $\sigma_{z,\max} = 5\,\mathrm{kPa}$ are not exceeded.

Task 2: Aorta under physiological blood pressure

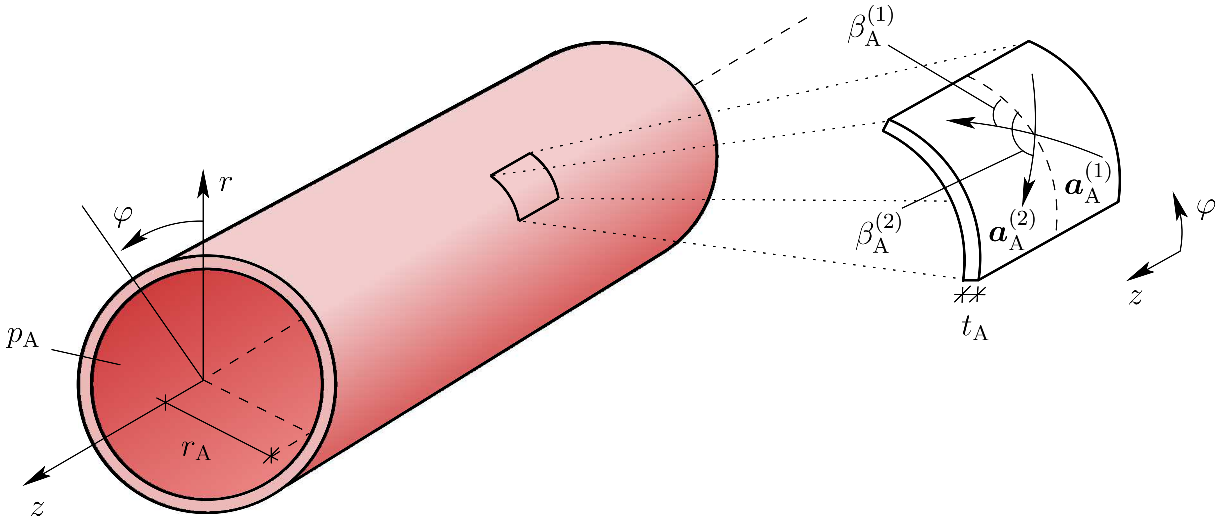


Figure 4: Illustration of a simplified healthy human aorta.

Now a thin-walled idealization of a healthy human aorta is considered, see Fig. 4. The two load-bearing fiber families are oriented symmetrically with respect to the axial direction. An internal blood pressure p_{A} is taken into account and strains in z-direction are again assumed to be zero, i. e. $\varepsilon_z = 0$. The whole set of parameters, where the index "A" indicates the artery, is given below.

$r_{\mathrm{A}} = 10\,\mathrm{mm}$,	$E_{\mathrm{A}} = 2\,\mathrm{kPa}$,	$\beta_{\mathrm{A}}^{(1)} = 30°$,
$t_{\mathrm{A}} = 1\,\mathrm{mm}$,	$\nu_{\mathrm{A}} = 0.49$,	$\beta_{\mathrm{A}}^{(2)} = 150°$.
$p_{\mathrm{A}} = 120\,\mathrm{mmHg} = 16\,\mathrm{kPa}$,	$\alpha_{\mathrm{A}} = \alpha_{\mathrm{A}}^{(1)} = \alpha_{\mathrm{A}}^{(2)} = 1000\,\mathrm{kPa}$,	

Due to the specific structural problem the same simplifications with respect to strains and stresses apply as for Task 1.

1. Here, the internal pressure is significantly higher and the fiber elasticity is significantly lower compared to the membrane material in Task 1. Before you start to mathematically calculate the values of σ_z and ε_φ try to estimate their values according to your experience obtained from Task 1.
2. Verify your assumptions by calculating ε_φ and σ_z applying the analogous procedure used in Task 1.
3. As we have seen, the arterial wall can bear relatively high pressure. Although technically unfeasible, would it in principle be possible to replace the textile membrane material in the roof construction from Task 1 by arterial tissue? For this purpose again consider the ultimate values given in Task 1.
4. Determine the fiber stiffness α_{A}^* that would be necessary to keep the circumferential strain of the arterial-tissue-made membrane lower than $\varepsilon_\varphi^* = 30\,\%$, which is considered to be a supportable value for the arterial material. Compare α_{A}^* with $\alpha_{\mathrm{M}}^{(1)}$ and $\alpha_{\mathrm{M}}^{(2)}$.